MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Absorption times. General CTMC. Matrix exponentials

Next: 6.6, Durrett 4.1

Week 3:

- HW2 due Friday, April 21 on Gradescope

Mean time until absorption

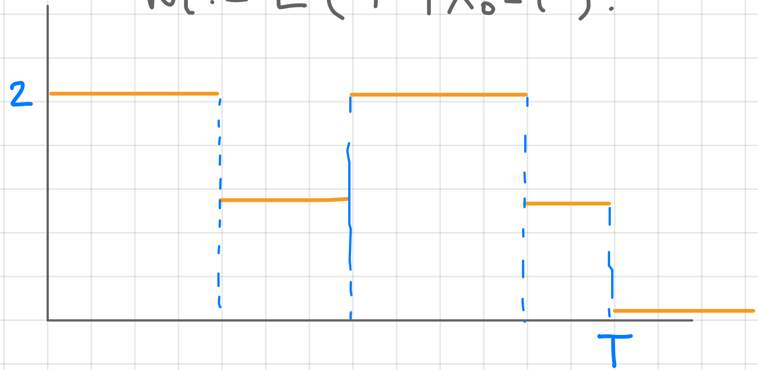
Let $(X_t)_{t\geq 0}$ be a birth and death process. Denote

$T = \min\{t \geq 0 : X_t = 0\}$ absorption time and

$w_i := E(T \mid X_0 = i)$.

Let $(Y_n)_{n\geq 0}$ be the jump chain for $(X_t)_{t\geq 0}$.

$$N := \min\{n \geq 0 : Y_n = 0\}$$



Then

$$w_i = E\left(\sum_{k=0}^{N-1} S_k \mid X_0 = i\right) = \frac{1}{\lambda_i + \mu_i} + E\left(\sum_{k=1}^{N-1} S_k \mid X_0 = i\right)$$

$$= \frac{1}{\lambda_i + \mu_i} + E\left(\sum_{k=1}^{N-1} S_k \mid X_0 = i, Y_1 = i+1\right) P(Y_1 = i+1 \mid Y_0 = i)$$

$$+ E\left(\sum_{k=1}^{N-1} S_k \mid X_0 = i, Y_1 = i-1\right) P(Y_1 = i-1 \mid Y_0 = i)$$

Mean time until absorption

$$\begin{cases} w_i = \dfrac{1}{\lambda_i+\mu_i} + \dfrac{\lambda_i}{\lambda_i+\mu_i} w_{i+1} + \dfrac{\mu_i}{\lambda_i+\mu_i} w_{i-1}, \\ w_0 = 0 \end{cases}$$

$$w_i = \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} + \sum_{k=1}^{i-1} \rho_k \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j \rho_j}, & \text{if } \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} < \infty \\ \infty, & \text{if } \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} = \infty \end{cases}$$

First step analysis for birth and death processes

Summary:

Let $(X_t)_{t\geq 0}$ be a birth and death process of rates $((\lambda_i,\mu_i))_{i\geq 0}$ with $\lambda_0 = 0$ (state 0 absorbing).

Denote $T = \min\{t : X_t = 0\}$, $u_i := P(X_t \text{ gets absorbed in } 0 \mid X_0 = i)$

$w_i = E(T \mid X_0 = i)$ and $\rho_j = \dfrac{\mu_1\mu_2\cdots\mu_j}{\lambda_1\lambda_2\cdots\lambda_j}$. Then

$$u_i = \begin{cases} \dfrac{\sum_{j=i}^{\infty}\rho_j}{1+\sum_{j=1}^{\infty}\rho_j}, & \text{if } \sum_{j=1}^{\infty}\rho_j < \infty \\ 1, & \text{if } \sum_{j=1}^{\infty}\rho_j = \infty \end{cases}$$

$$w_i = \begin{cases} \sum_{j=1}^{\infty}\dfrac{1}{\lambda_j\rho_j} + \sum_{k=1}^{i-1}\rho_k\sum_{j=k+1}^{\infty}\dfrac{1}{\lambda_j\rho_j}, & \text{if } \sum_{j=1}^{\infty}\dfrac{1}{\lambda_j\rho_j} < \infty \\ \infty, & \text{if } \sum_{j=1}^{\infty}\dfrac{1}{\lambda_j\rho_j} = \infty \end{cases}$$

Birth and death processes. Results

- infinitesimal transition probability description
- sojourn time description (jump and hold)

  sojourn times are independent exponential r.v.s

  $P(i \to i+1) = \frac{\lambda_i}{\lambda_i + \mu_i}$, $P(i \to i-1) = \frac{\mu_i}{\lambda_i + \mu_i}$

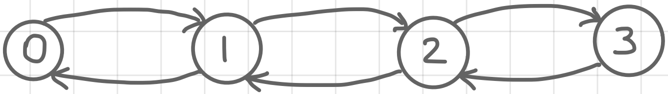
- system of differential equations for pure birth/death

  e.g. $P_i'(t) = -\lambda_i P_i(t) + \lambda_{i-1} P_{i-1}(t)$

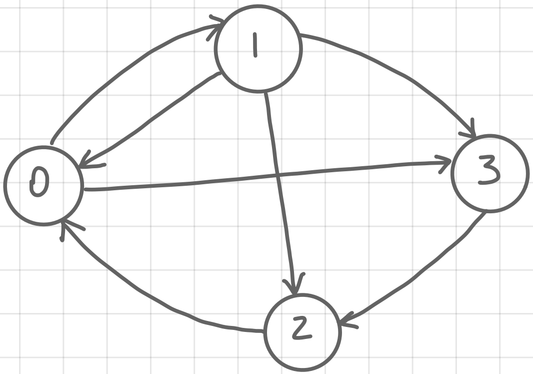
- distributions of X_t for linear birth (geometric) and linear death (binomial) processes
- first step analysis giving absorption probabilities and mean time to absorption
- explosion, Strong Markov property etc.

General continuous time MC

Assume for simplicity that the state space is finite



birth and death process



general MC

How to define? How to analyze?

Q-matrices (infinitesimal generators)

Let $S = \{0, 1, \dots, N\}$. We call $Q = (q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a)

(b)

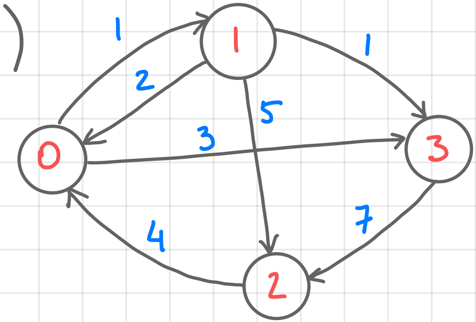
(c)

Examples

(a)

$$Q = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

(b)



$$\begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series

converges componentwise, and we denote

its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =:$, the matrix exponential of Q.

In particular, we can define for $t \geq 0$.

<u>Thm</u>. Define $P(t) = e^{tQ}$. Then

(i) for all s, t

(ii) $(P(t))_{t\geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases} , \quad \text{and} \quad \begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}$$

Matrix exponentials

Properties are easy to remember $\longrightarrow$ scalar exponential

(i) $e^{(t+s)Q} = e^{tQ} e^{sQ} = e^{sQ} e^{tQ}$ $\quad \left(e^{(t+s)\alpha} = e^{t\alpha} e^{s\alpha} \right)$

(note that in general $AB \neq BA$ for matrices A, B)

(ii) $\frac{d}{dt} e^{tQ} = Q e^{tQ} = e^{tQ} Q$ $\quad \left(\frac{d}{dt} e^{t\alpha} = \alpha e^{t\alpha} \right)$

$e^{0 \cdot Q} = I$ $\quad (e^0 = 1)$

Example

(a) $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(b) $Q_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Matrix exponentials

Results on the previous slide hold for any matrix Q.

Thm. Matrix Q is a Q-matrix

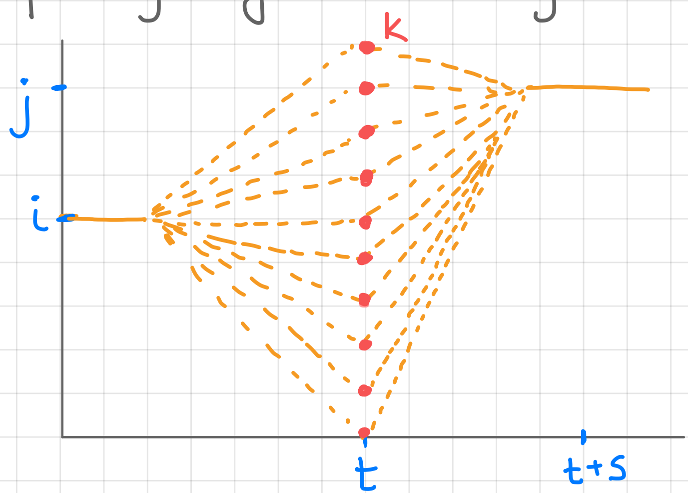
iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

Remarks The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

(if you think about MC →

Chapman-Kolmogorov)



Main theorem

Let $P(t)$ be a matrix-valued function $t \geq 0$.
Consider the following properties

(a) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ for all $i, j, t \geq 0$

(b) $P(0) = I$

(c) $P(t+s) = P(t)P(s)$ for all $t, s \geq 0$

(d) $\lim_{t \downarrow 0} P(t) = I$ (continuous at 0)

<u>Theorem A.</u> $P(t)$ satisfies (a)–(d)

if and only if

Main theorem. Remarks

This theorem establishes one-to-one correspondance between matrices $P(t)$ satisfying (a)-(d) and the Q-matrices of the same dimension.

Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable

2. If $P(t) = e^{tQ}$, then $P(h) =$ as $h \to 0$

$$P(h) =$$

Q-matrices and Markov chains

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots,N\}$ with right-continuous sample paths

Denote $P_{ij}(t) = P(X_t = j \mid X_0 = i)$, $i,j \in \{0,1,\dots,N\}$

stationary

Then

- $P_{ij}(t)$, $\sum_{j=0}^{N} P_{ij}(t)$ $\left(= \sum_{j=0}^{N} P(X_t = j \mid X_0 = i) \right)$
- $P_{ij}(0) =$ $\left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \right)$
- $P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i)$
- $\lim_{h \downarrow 0} P(X_h = j \mid X_0 = i) =$

Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix Q such that

$$P(t) =$$

In particular,

$$P(h) =$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

Q is called the infinitesimal generator of $(X_t)_{t\geq 0}$

Infinitesimal description of cont. time MC

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix, let $(X_t)_{t\geq 0}$ be right-continuous stochastic process, $X_t \in \{0, 1, \dots, N\}$.

We call $(X_t)_{t\geq 0}$ a Markov chain with generator Q, if

(i) $(X_t)_{t\geq 0}$ satisfies the Markov property

(ii) $P(X_{t+h} = j \mid X_t = i) =$

Example

Pure death process

- $P_{i,i-1}(h) = \mu_i h + o(h)$
- $P_{ii}(h) = 1 - \mu_i h + o(h)$
- $P_{ij}(h) = o(h)$ for $j \notin \{i-1, i\}$

The corresponding Q-matrix

$$Q = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j \neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} & q_{01} & q_{02} & \cdots \\ q_{10} & & q_{12} & \cdots \\ q_{20} & q_{21} & & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$q_0 = \sum_{i \neq 0} q_{0i}$

$\vdots$

Denote $Y_k := X_{W_k}$ (jump chain).

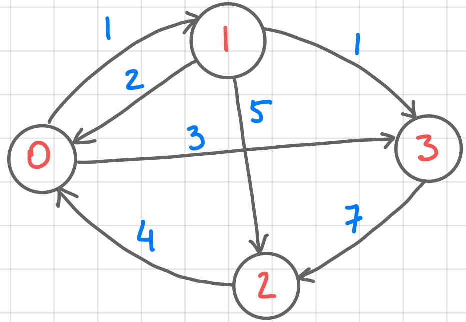
Then the MC with generator matrix Q has the following equivalent jump and hold description

- sojourn times S_k are independent r.v. with $P(S_k > t \mid Y_k = i) =$

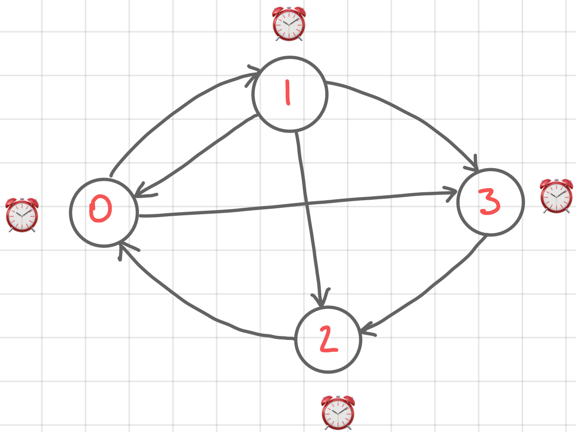
- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) =$

Example

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & & & & \\
1 & & & & \\
2 & & & & \\
3 & & & &
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -7 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix}
\end{array}
$$

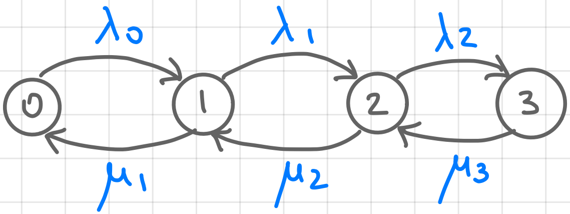


$i \xrightarrow{\alpha} j$ "="



Example

Birth and death process on $\{0,1,2,3\}$



$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \mu_3 & -\mu_3 \end{pmatrix}$$