MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: General CTMC. Matrix FSA for general MC

Next: PK 6.3, 6.6, Durrett 4.2

Week 3:

- HW2 due Friday, April 21 on Gradescope
- No in-person lecture on Friday, April 21

Matrix exponentials

Results on the previous slide hold for any matrix Q.

Thm. Matrix Q is a Q-matrix

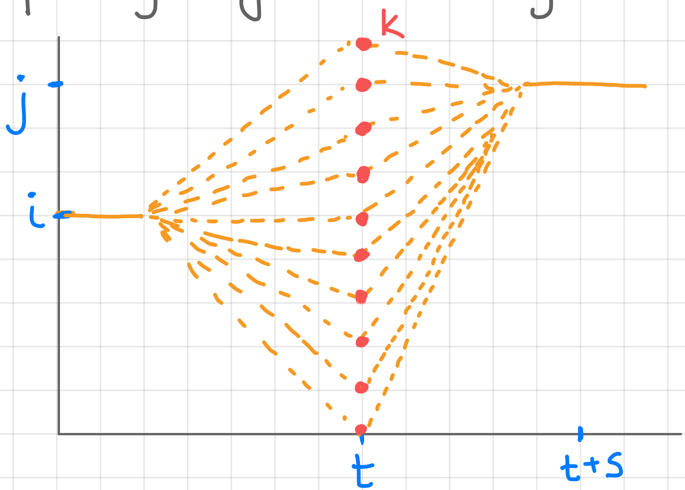
iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

$$P_{ij}(t) \geq 0, \qquad \sum_j P_{ij}(t) = 1 \quad \forall i \text{ and } \forall t \geq 0$$

Remarks The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

$$\sum_{k=0}^{N} P_{ik}(t) P_{kj}(s)$$



(if you think about MC $\rightarrow$

Chapman-Kolmogorov)

Main theorem

Let $P(t)$ be a matrix-valued function $t \geq 0$.

Consider the following properties

(a) $P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ for all $i, j, t \geq 0$

(b) $P(0) = I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \quad 1 \end{pmatrix}$

(c) $P(t+s) = P(t)P(s)$ for all $t, s \geq 0$

(d) $\lim_{t \downarrow 0} P(t) = I$ (continuous at 0)

<u>Theorem A.</u> $P(t)$ satisfies (a)–(d)

if and only if

$P(t) = e^{tQ}$ for some Q-matrix Q

Main theorem. Remarks

This theorem establishes a one-to-one correspondance between matrices $P(t)$ satisfying (a)-(d) and the Q-matrices of the same dimension.

Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable

2. If $P(t) = e^{tQ}$, then $P(h) = I + Qh + o(h)$ as $h \to 0$

$$P(h) = I + hQ + \underbrace{\sum_{k=2}^{\infty} \frac{Q^k h^k}{k!}}_{o(h)}$$

Q-matrices and Markov chains

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots,N\}$ with right-continuous sample paths

Denote $P_{ij}(t) = P(X_t = j \mid X_0 = i)$, $i,j \in \{0,1,\dots,N\}$

stationary

Then

- $P_{ij}(t) \geq 0$, $\sum_{j=0}^{N} P_{ij}(t) = 1$ $\left(= \sum_{j=0}^{N} P(X_t = j \mid X_0 = i)\right)$
- $P_{ij}(0) = \delta_{ij}$ $\left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \right)$
- $P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i) = \sum_{k=0}^{N} P_{kj}(s) P_{ik}(t)$

$$= \sum_{k=0}^{N} P(X_{t+s} = j \mid X_0 = i, X_t = k) P(X_t = k \mid X_0 = i)$$

- $\lim_{h \downarrow 0} P(X_h = j \mid X_0 = i) = \delta_{ij}$

Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix Q such that

$$P(t) = e^{tQ}$$

In particular,

$$P(h) = I + Qh + o(h)$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

Q is called the infinitesimal generator of $(X_t)_{t\geq 0}$

Infinitesimal description of cont. time MC

Let $Q=(q_{ij})_{i,j=0}^{N}$ be a Q-matrix, let $(X_t)_{t\geq 0}$ be right-continuous stochastic process, $X_t \in \{0,1,\dots,N\}$.

We call $(X_t)_{t\geq0}$ a Markov chain with generator Q, if

(i) $(X_t)_{t\geq0}$ satisfies the Markov property

(ii) $$P(X_{t+h}=j \mid X_t=i)=\begin{cases} q_{ij}h+o(h) & \text{if } i\neq j \\ 1+q_{ii}h+o(h) & \text{if } i=j \end{cases}$$

Example

Pure death process

- $P_{i,i-1}(h)=\mu_i h+o(h)$
- $P_{ii}(h)=1-\mu_i h+o(h)$
- $P_{ij}(h)=o(h)$ for $j\notin\{i-1,i\}$

The corresponding Q-matrix

$$Q=\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N \end{array}\begin{pmatrix} 0 & 0 & 0 & \cdots & & 0 \\ \mu_1 & -\mu_1 & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 & -\mu_2 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & \cdots & & 0 & \mu_N & -\mu_N \end{pmatrix}$$

Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j\neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} -q_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -q_1 & q_{12} & \cdots \\ q_{20} & q_{21} & -q_2 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix} \qquad q_0 = \sum_{i\neq 0} q_{0i} \quad \vdots$$

Denote $Y_k := X_{W_k}$ (jump chain).

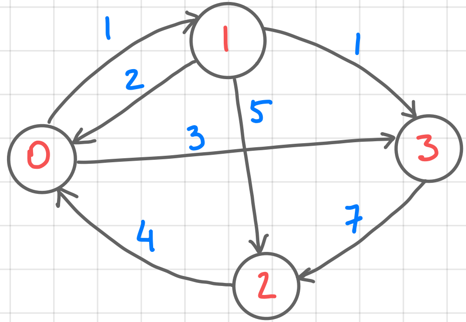
Then the MC with generator matrix Q has the following equivalent jump and hold description

- sojourn times S_k are independent r.v. with $P(S_k > t \mid Y_k = i) = e^{-q_i t}$ $\left(S_k \sim Exp(q_i) \right)$
- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) = \dfrac{q_{ij}}{q_i}$

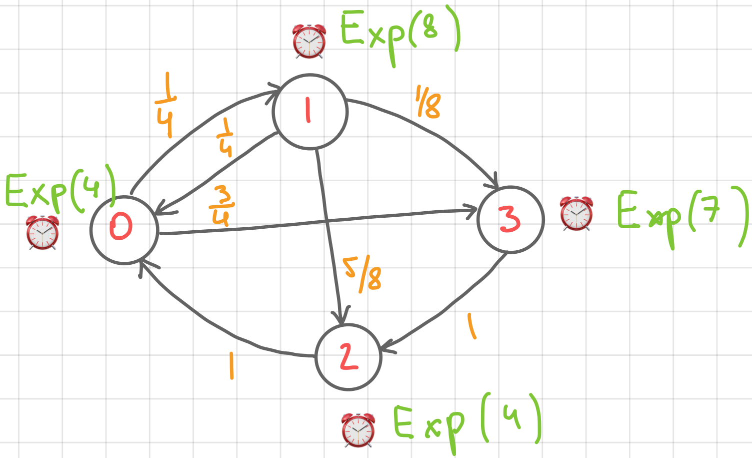
Example

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{array}$$

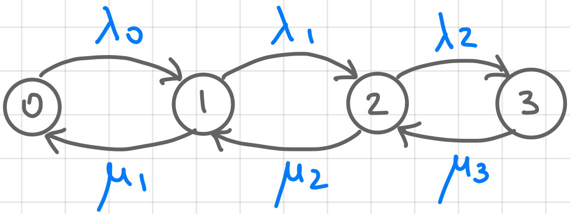


"=" $P_{ij}(h) = \alpha h + o(h)$



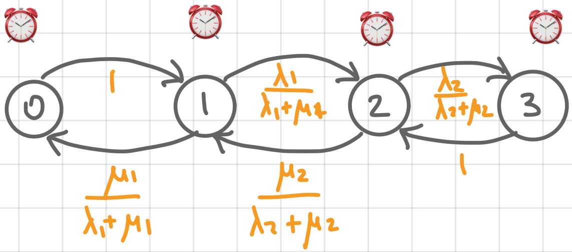
Example

Birth and death process on $\{0,1,2,3\}$



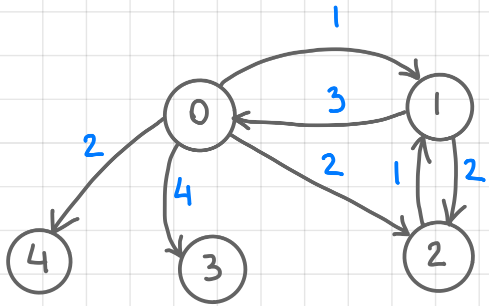
$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \mu_3 & -\mu_3 \end{pmatrix}$$

$\mathrm{Exp}(\lambda_0)$ $\mathrm{Exp}(\lambda_1+\mu_1)$ $\mathrm{Exp}(\lambda_2+\mu_2)$ $\mathrm{Exp}(\mu_3)$



General continuous time finite state MCs

Rate diagram



Generator

$$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -9 & 1 & 2 & 4 & 2 \\ 1 & 3 & -5 & 2 & & \\ 2 & & 1 & -1 & & \\ 3 & & & & 0 & \\ 4 & & & & & 0 \end{array}$$

Infinitesimal description

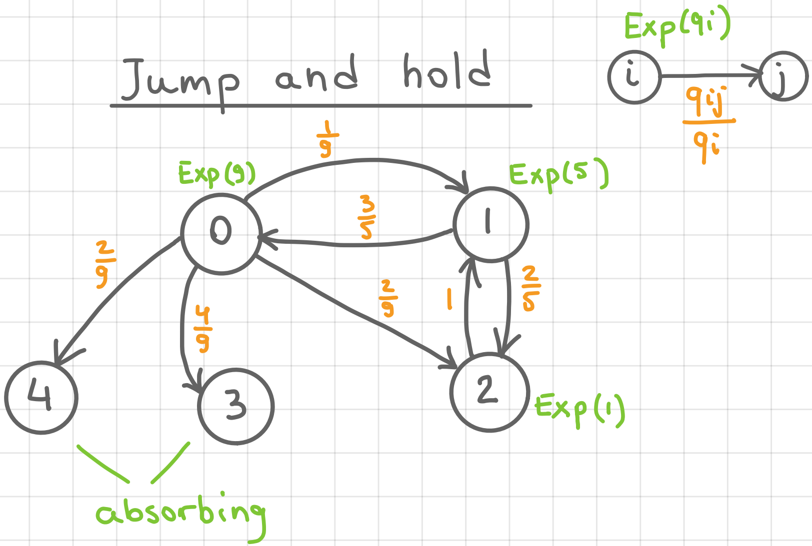
$$P_{ij}(h) = q_{ij}h + o(h) \quad , i \neq j$$

$$P_{ii}(h) = 1 - q_i h + o(h)$$

$$P_{02}(h) = 2h + o(h)$$

$$P_{00}(h) = 1 - 9h + o(h)$$

Jump and hold



Absorption probabilities for finite state chains

By considering the jump chain $(Y_n)_{n \geq 0}$ with $Y_n = X_{W_n}$ and its transition probabilities $P(Y_{n+1} = j \mid Y_n = i) = \frac{q_{ij}}{q_i}$ we can apply the first step analysis to compute, e.g., the absorption probabilities (similarly as for B&D)

If state i is absorbing, then $q_{ij} = 0$ for all $j \neq i$ (no jumps from state i), so $q_i = q_{ii} = 0$. Let Q be given by

$$
Q = \begin{array}{cc}
 & \begin{array}{cccccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\
\begin{array}{c} 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array} &
\left(\begin{array}{ccc|ccc}
-q_0 & & & \cdots & q_{ij} & \\
\vdots & & & & \vdots & \\
q_{ij} & \cdots & -q_{K-1} & & \vdots & \\
\hline
 & & & 0 & & 0 \\
 & 0 & & & \ddots & \\
 & & & & & 0
\end{array}\right)
\end{array}
$$

with $\{0, \dots, K-1\}$ transient,

$\{K, \dots, N\}$ absorbing

Absorption probabilities for finite state chains

$$
Q = \begin{array}{c|cccc|ccc}
 & 0 & \cdots & & K-1 & K & \cdots & N \\
\hline
0 & -q_0 & & & & \cdots & q_{ij} & \\
\vdots & \vdots & & & & & \vdots & \\
K-1 & q_{ij} & \cdots & & -q_{K-1} & & & \\
\hline
K & & & & & 0 & & 0 \\
\vdots & & & 0 & & & \ddots & \\
N & & & & & & & 0
\end{array}
$$

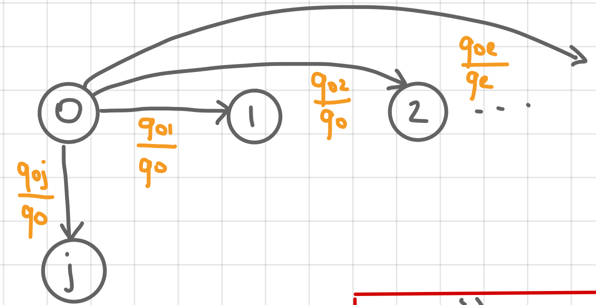
Let $i \in \{0, \dots, K-1\}$, $j \in \{K, \dots, N\}$.

Let $\mathbf{M} = \min\{n : Y_n \in \{K, \dots, N\}\}$

Denote $u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid X_0 = i)$.

Then FSA leads to the system

Jump chain



$$u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid Y_0 = i)$$

$$=$$

$$
\boxed{u_i^{(j)} = \frac{q_{ij}}{q_i} + \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \frac{q_{i\ell}}{q_i} u_\ell^{(j)}}
$$

$P(Y_{n+1} = j \mid Y_n = i)$ $\qquad$ $P(Y_{n+1} = \ell \mid Y_n = i)$