☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

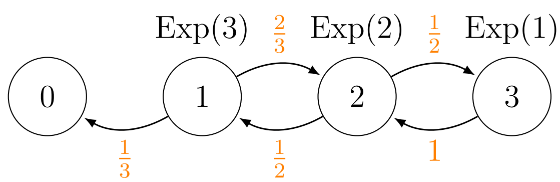
☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

1. (30 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3\}$ with state 0 absorbing, birth rates $\lambda_1 = 2$, $\lambda_2 = 1$ and the death rates $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$.
   (a) Draw the diagram of the jump chain of $(X_t)_{t\geq 0}$ and indicate the distribution of the sojourn times.
   (b) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on the set $\{1,2,3\}$. Compute the expectation of the time at which the process is absorbed in state 0.

Solution.

(a) The diagram of the jump chain of $(X_t)_{t\geq 0}$ has the following form



The probability $P_{i,i+1}$ (the probability of jumping from state i to state $i+1$) is equal to

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i},$$

and, similarly,

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}.$$

The sojourn time in state i has exponential distribution with rate $\lambda_i + \mu_i$.

(b) Denote by v_i the expected time to absorption given that $X_0 = i$, $i \in \{1,2,3\}$. Then, using the first step analysis, v_1, v_2, v_3 satisfy the following system of equations

$$\begin{aligned} v_1 &= \frac{1}{3} + \frac{2}{3}v_2, \\ v_2 &= \frac{1}{2} + \frac{1}{2}v_1 + \frac{1}{2}v_3, \\ v_3 &= 1 + v_2. \end{aligned}$$

Substituting the first and the third equations into the second, we get

$$\begin{aligned} v_2 &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} + \frac{2}{3}v_2\right) + \frac{1}{2}(1 + v_2), \\ v_2 &= \frac{7}{6} + \frac{5}{6}v_2, \\ v_2 &= 7, \quad v_3 = 8, \quad v_1 = 5. \end{aligned}$$

Using the law of total probability, the average time to absorption in state 0 is equal to

$$\frac{1}{3}v_1 + \frac{1}{3}v_2 + \frac{1}{3}v_3 = \frac{1}{3}(7+8+5) = \frac{20}{3}.$$

2. (30 points) Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on the state space $\{0,1,2\}$ with transition probability functions

$$P(t) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\left\|\begin{array}{c|c|c} 0 & 1 & 2 \\ \frac{1}{6}+\frac{3}{2}e^{-4t}-\frac{2}{3}e^{-3t} & \frac{1}{6}-\frac{3}{2}e^{-4t}+\frac{4}{3}e^{-3t} & \frac{2}{3}-\frac{2}{3}e^{-3t} \\ \frac{1}{6}+\frac{1}{2}e^{-4t}-\frac{2}{3}e^{-3t} & \frac{1}{6}-\frac{1}{2}e^{-4t}+\frac{4}{3}e^{-3t} & \frac{2}{3}-\frac{2}{3}e^{-3t} \\ \frac{1}{6}-\frac{1}{2}e^{-4t}+\frac{1}{3}e^{-3t} & \frac{1}{6}+\frac{1}{2}e^{-4t}-\frac{2}{3}e^{-3t} & \frac{2}{3}+\frac{1}{3}e^{-3t} \end{array}\right\|.$$

(a) Determine the distribution of the sojourn times of the process in states 0, 1 and 2.

(b) In the long run, what fraction of time will the process $(X_t)_{t\geq 0}$ spend in state 0? [**Hint.** You can answer this question without solving any equations, and if you do so you should clearly state which results you use.]

(c) Let $Q = (q_{ij})_{i,j=0}^2$ be the generator matrix of $(X_t)_{t\geq 0}$. Compute q_{10}. Suppose you observe the process jumping from state 2 to state 0. What is the average time that you have to wait until the next time you observe the jump from state 2 to state 0?

Solution.

(a) The distribution of the sojourn times can be read off from the infinitesimal generator Q, and from the relation between the Markov semigroup $P(t)$ and Q we have that $Q = P'(0)$. Therefore, to determine the distribution of the sojourn times it is enough to compute the derivatives of the diagonal entries of $P(t)$ at $t = 0$

$$P'_{00}(0) = -4, \quad P'_{11}(0) = -2, \quad P'_{22}(0) = -1.$$

Thus, the sojourn times in states 0, 1 and 2 have exponential distributions with rates $q_0 = 4$, $q_1 = 2$, $q_2 = 1$ correspondingly.

(b) Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the stationary distribution for the Markov chain $(X_t)_{t\geq 0}$. Then π_i, $i \in \{0,1,2\}$, gives the average long run fraction of time spent by the process in state i.

In order to compute π_0, note that from the theorem about the long run behavior of continuous time Markov chains, $P_{i0} \to \pi_0$ as $t \to \infty$. If we take the limit in the above explicit formula for $P(t)$ we get

$$\lim_{t\to\infty} P(t) = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}, \tag{1}$$

and thus on average in the long run the process spends 1/6 of the time in state 0.

(c) If Q is the infinitesimal generator of $(X_t)_{t\geq 0}$, then

$$q_{10} = P'_{10}(0) = 0.$$

In particular this means that the process cannot jump directly from state 1 to state 0; the process can jump to state 0 only from state 2.

In order to compute the average time required to observe the transition from 2 to 0 happening again, we can either apply the first step analysis, or use the theorem about the long run behavior of the continuous time Markov chains. I present below the second solution.

From the theorem about the long run behavior of the continuous time Markov chains,

$$\pi_i = \frac{1}{q_i m_i},$$

where m_i is the average return time to state i. From this we have that the average return time to 0 is given by

$$m_0 = \frac{1}{q_0 \pi_0} = \frac{1}{4 \cdot \frac{1}{6}} = \frac{3}{2}.$$

If you observe the transition from state 2 to state 0, then the return of the process to state 0 can only occur through a transition from 2 to 0 ($q_{10} = 0$, so the jumps from 1 to 0 are not allowed). Therefore, the average time to see again the transition from 2 to 0 is equal to $m_0 = \frac{3}{2}$.

3. (30 points) Certain printing facility has two printers operating on a 24/7 basis and one repairman that takes care of the printers. The amount of time (in hours) that a printer works before breaking down has exponential distribution with mean 2. If a printer is broken, the repairman needs exponentially distributed amount of time with mean 1 (hour) to repair the broken printer. The repairman cannot repair two printers simultaneously. Each printer can produce 100 pages per minute.

   Let X_t denote the number of printers in operating state at time t.

   (a) Assuming without proof that $(X_t)_{t\geq 0}$ is a Markov process, determine the generator of $(X_t)_{t\geq 0}$ (you can provide rigorous computations for only one entry of matrix Q.) [Hint. If $T \sim \text{Exp}(\gamma)$, then $P(T \leq h) = \gamma h + o(h)$ as $h \to 0$.]

   (b) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

   (c) In the long run, how many pages does the facility produce on average per minute?

Solution.

(a) The generator of $(X_t)_{t\geq 0}$ is given by

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} -1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 0 & 1 & -1 \end{array} \right\| \end{array}. \tag{2}$$

Example of computations of q_{ij}.

– $X_0 = 1$ means that one printer is operating and one printer is broken.

  • $X_h = 0$ means that the operating printer stops working before time h and the broken printed is not repaired before time h, so

$$\begin{aligned} &P(X_h = 0 \mid X_0 = 1) && (3)\\ &= P(\text{printer's working time} \leq h)P(\text{printer's repair time} > h) + o(h) && (4)\\ &= (1 - e^{-\frac{1}{2}h})e^{-h} + o(h) = \frac{1}{2}h + o(h) && (5) \end{aligned}$$

  and $q_{10} = \frac{1}{2}$.

  • $X_h = 2$ means that the broken printer is repared before time h and the operating printer is still working at time h, so

$$\begin{aligned} P(X_h = 2 \mid X_0 = 1)& && (6)\\ &= P(\text{repare time} \leq h)P(\text{working time} > h) + o(h) && (7)\\ &= (1 - e^{-h})e^{-\frac{1}{2}h} + o(h) = h + o(h) && (8) \end{aligned}$$

  and $q_{12} = 1$.

- $X_0 = 2$ means that both printers are working
  - $X_h = 1$ means that one of the two printers stops working before time h and the other is working at time h (note that there are two choices of which of the two is broken), so

$$
\begin{aligned}
&P(X_h = 1 \,|\, X_0 = 2) &(9)\\
&= 2P(\text{printer's working time} \le h)P(\text{printer's working time} > h) + o(h) &(10)\\
&= 2(1 - e^{-\frac{1}{2}h})e^{-\frac{1}{2}h} + o(h) = 2\Big(\frac{1}{2}h + o(h)\Big) = h + o(h) &(11)
\end{aligned}
$$

and $q_{21} = 1$.

(b) Let (π_0, π_1, π_2) be the stationary distribution. Then (π_0, π_1, π_2) satisfies the following system

$$
\begin{aligned}
-\pi_0 + \frac{1}{2}\pi_1 &= 0, &(12)\\
\pi_0 - \frac{3}{2}\pi_1 + \pi_2 &= 0, &(13)\\
\pi_1 - \pi_2 &= 0, &(14)\\
\pi_0 + \pi_1 + \pi_2 &= 1. &(15)
\end{aligned}
$$

From the first and the third equations we have that $\pi_0 = \frac{1}{2}\pi_1$ and $\pi_2 = \pi_1$. Plugging this into the last equation gives

$$\pi_1\Big(\frac{1}{2} + 1 + 1\Big) = \frac{5}{2}\pi_1 = 1, \qquad (16)$$

from which we get that

$$\pi_0 = \frac{1}{5}, \quad \pi_1 = \frac{2}{5}, \quad \pi_2 = \frac{2}{5}. \qquad (17)$$

(c) In the long run, $\frac{1}{5}$ of the time both printers are broken printing 0 pages per minute, $\frac{2}{5}$ of the time only one printer is working producing 100 pages per minute and $\frac{2}{5}$ of the time both printer are working producing 200 pages per minute. Therefore, on average the printing facility produces

$$\frac{1}{5} \cdot 0 + \frac{2}{5} \cdot 100 + \frac{2}{5} \cdot 200 = 120 \qquad (18)$$

pages per minute.